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FORMULAE FUNDAMENTALES
GEOMETRIAE ANALYTICAE SPHAERICAE

IN USU

ANGULORUM COORDINATORUM EVOLUTAE.

PARS I.

DISSERTATIO

MATHEMATICA INAUGURALIS,

QUAM UNA CUM THESIBUS CONTROVERSIIS

CONSENSU ET AUCTORITATE

AMPLISSIMI PHILOSOPHORUM ORDINIS MONASTERIENSIS

AD

SUMMOS IN PHILOSOPHIA HONORES

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IGNATIUS PHILIPPUS RENVERS

TELGETANUS GUESTPHALUS

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Reciprocitatis sphæricæ methodus simplicissima et maxime usitata hoc modo exprimi potest:

„Cum quovis circulo maximo punctum, quod ejus centrum sphæricum est, cum quovis puncto circulus maximus, ex hoc puncto descriptus reciprocitatis sphæricæ lege conjunctus est.“

Hac lege ad usitata systemata quantitatum coordinatarum geometriæ analyticæ sphæricæ adhibita formulæ, quæ sequuntur, fundamentales deductæ sunt.

§. 1.

Sit (fig. 1.) XVY triangulum sphæricum, terminatum ergo tribus arcubus circulorum maximorum sphæræ in ejus superficie mutuo se intersecantium, præterea $VX = VY = \frac{\pi}{2}$ (si sphæræ radius $= 1$ est), denique XY cujuslibet longitudinis. Arcus VX et VY *axes*, angulos VXY et VYX *axium angulos*, arcum XY *lineam cardinalem*, punctum V *punctum cardinale* nominamus.

Situm circuli alicujus maximi, qui axes in punctis A et B secet, variis modis determinare possumus. Si puncta X et B, Y et A arcubus circulorum maximorum conjungimus, circulus maximus (AB) determinatus est

$$\text{angulis } XYA = x \text{ et } YXB = y,$$

dummodo utramque axem secet. Vocamus hos angulos *axium angulos coordinatos.* $XYA = x$ est æquatio puncti A, $XYB = y$ æquatio puncti B, altera æquationum $x = \frac{\pi}{2}$ et $y = \frac{\pi}{2}$ est æquatio puncti cardinalis V. Circulus maximus YA determinatur angulis $y = o$, $x = XYA$; circulus maximus XB angulis $x = o$, $y = YXB$; circulus maximus YX angulis $x = y = o$; circulus maximus YV angulis $x = \frac{\pi}{2}$, $y = o$; circulus maximus XV angulis $y = \frac{\pi}{2}$, $x = o$. Omnes ceteros circulos per punctum cardinale V ductos hoc modo determinari non posse, manifestum est; determinantur autem divisionis ratione aut anguli XVY ipsius, aut ejus supplementi.

Deinde circulus maximus (AB) determinari potest

$$\text{angulo } XYA = x \text{ et fractione } \frac{\sin BAY}{\sin BAX} = \frac{\sin \alpha}{\sin(\alpha + \beta)}$$

$$\text{vel angulo } YXB = y \text{ et fractione } \frac{\sin ABX}{\sin ABY} = \frac{\sin \gamma}{\sin(\gamma + \delta)},$$

si ponimus $\angle BAY = \alpha$, $YAX = \beta$, $ABX = \gamma$, $XBY = \delta$.

Etiam angulis α et γ, quos *angulos applicatos* nominamus, situs circuli maximi (AB) determinari potest; his vero angulis non uteris, nisi sit $XY = \frac{\pi}{2}$.

Arcum $XA = X$ *amplitudinem priorem*, arcum $YB = Y$ *amplitudinem secundam* circuli maximi (AB) nominamus. His quoque circulus noster determinatur.

Postremo arcus $YA = p$ et $XB = q$ *arcus applicati*, arcus AB circuli determinandi arcus lineæ cardinali oppositus, vel simpliciter *arcus oppositus* nominentur.

Adnot. 1. Omnibus his quantitatibus circulus non determinatur, si per punctum cardinale ductus est.

Adnot. 2. Si linea cardinalis $XY = \frac{\pi}{2}$ est, est $\beta = \delta = \frac{\pi}{2}$, $p = q = \frac{\pi}{2}$; $X = x$, $Y = y$; $\frac{\sin\,\alpha}{\sin\,(\alpha + \beta)} = \text{tang}\,\alpha$, $\frac{\sin\,\gamma}{\sin\,(\gamma + \delta)} = \text{tang}\,\gamma$.

Quum sit XY mensura anguli XVY, in sequentibus ponemus $XY = v$.

§. 2.

Nexus harum quantitatum simplicissimis hisce formulis exprimitur:

1. $\cot p = \cot v.\cos x$; $\cot q = \cot v.\cos.\,y$.

2. $\sin\,\beta = \frac{\sin\,v}{\sin\,p}$; $\sin\,\delta = \frac{\sin\,v}{\sin\,q}$.

3. $\cos\,\beta = \cos\,v.\sin\,x$; $\cos\,\delta = \cos\,v.\sin\,y$.

4. $$\begin{cases} \sin X = \sin p.\sin x; \ \sin.\,Y = \sin q.\sin y. \\ \cos X = \frac{\cos p}{\cos v}; \ \cos Y = \frac{\cos q}{\cos v}. \end{cases}$$

5. $\text{tang}\,X = \sin v.\,\text{tang}\,x$; $\text{tang}\,Y = \sin v.\,\text{tang}\,y$.

Quum sit $\frac{\sin(\alpha+\beta)}{\sin\alpha} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha} = \cos\beta + \cot\alpha.\sin\beta$, et $\cot\alpha = \frac{\cos Y \sin p - \sin Y \cos p \sin x}{\sin y.\,\cos x} = \cot Y \frac{\sin p}{\cos x} - \cos p.\,\text{tang}\,x$ *), habemus, si formulæ 1., 2., 3., 5. adhibentur,

6. $\frac{\sin\,\alpha}{\sin\,(\alpha + \beta)} = \text{tang}\,y.\cos x$, et eodem modo: $\frac{\sin\,\gamma}{\sin\,(\gamma + \delta)} = \text{tang}\,x.\cos y$.

Adnot. Si linea cardinalis $XY = v = \frac{\pi}{2}$ est, habemus:

7. $\text{tang}\,\alpha = \text{tang}\,y\,.\cos x$; $\text{tang}\,\gamma = \text{tang}\,x\,.\cos y$.

8. $\text{tang}\,x = \frac{\text{tang}\,\gamma}{\cos\,y}$; $\sin\,\alpha = \sin\,y\,.\cos\,\gamma$ **).

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*) cf. Lehrbuch der niederen Sphärik von Dr. Chr. Gudermann, Münster 1835, §. 129.

**) cf. ibidem §. 123, 1.

Formulæ 8. transmutari possunt in:

$$\sin^2 x = \frac{\tang^2 \gamma}{\tang^2 \gamma + \cos^2 y}; \quad \tang^2 \alpha = \frac{\sin^2 y}{\tang^2 \gamma + \cos^2 y},$$

quæ formulæ propter denominatorum identitatem dignæ sunt, quæ animadvertantur.

§. 3.

(Fig. 2.) Situs circuli maximi, qui, ut antea, axes secet in punctis A et B, præterea determinari potest angulo, quem cum circulo maximo, cujus pars est linea cardinalis, facit, et ratione divisionis lineæ cardinalis.

Sit $CY = \alpha$, $CX = \beta$, $< AYX = x$, $BXY = y$; $XA = X$, $YB = Y$, $< ACX = z$, $XY = v$.

Tum est: $\tang Y = \tang z \,.\, \sin \alpha$, et quum ex formula 5. §. 2. sit $\tang Y = \tang y \sin v$, habemus formulam:

$$\frac{\tang y}{\tang z} = \frac{\sin \alpha}{\sin v} \text{ et similiter inventam } \frac{\tang x}{\tang z} = \frac{\sin \beta}{\sin v},$$

ex quibus facile invenitur:

1. $\tang z = \sqrt{[\tang^2 x - 2 \tang x \,.\, \tang y \cos v + \tang^2 y]}$.

2. $\sin \alpha = \frac{\sin v \,.\, \tang y}{\tang z}; \quad \sin \beta = \frac{\sin v \,.\, \tang x}{\tang z}.$

3. $\cos \alpha = \frac{\tang x - \cos v \,.\, \tang y}{\tang z}; \quad \cos \beta = \frac{\tang y - \cos v \,.\, \tang x}{\tang z}.$

4. $\tang \alpha = \frac{\sin v \,.\, \tang y}{\tang x - \cos v \,.\, \tang y}; \quad \tang \beta = \frac{\sin v \,.\, \tang x}{\tang y - \cos v \,.\, \tang x}.$

In his formulis ponimus x pro tang x, y pro tang y, z pro tang z, quo fiunt:

$$z = \sqrt{[x^2 - 2xy \cos v + y^2]}; \quad \sin \alpha = \frac{y \,.\, \sin v}{z}; \quad \cos \alpha = \frac{x - y \cos v}{z}; \quad \sin \beta = \frac{x \,.\, \sin v}{z};$$

$$\cos \beta = \frac{y - x \cos v}{z}; \quad \tang \alpha = \frac{y \sin v}{x - y \cos v}; \quad \tang \beta = \frac{x \,.\, \sin v}{y - x \cos v}.$$

Addimus præterea $\frac{1}{\cos z} = \sqrt{[1 + x^2 + y^2 - 2xy \cos v]}$.

Adnot. In sequentibus ubique x pro tang x, y pro tang y, z pro tang z positum est.

Longitudo arcus AB, intercepti ab axibus, ab angulis axium coordinatis pendet. Est enim:

$\cos AB = \cos Y \cos X \cos v + \sin Y \sin X = \cos Y \cos X (\cos v + \tang Y \tang X)$, ergo:

5. $\cos AB = \frac{\cos v + xy \sin^2 v}{\sqrt{(1 + x^2 \sin^2 v)} \,.\, \sqrt{(1 + y^2 \sin^2 v)}}; \quad \sin AB = \frac{\sin v \,.\, \sqrt{[1 + x^2 + y^2 - 2xy \cos v]}}{\sqrt{(1 + x^2 \sin^2 v)} \,.\, \sqrt{(1 + y^2 \sin^2 v)}}.$

Si $AB = \frac{\pi}{2}$ est, est ergo $xy = -\frac{\cos v}{\sin^2 v}$.

Formulis 5. ad deducendum differentiale perimetri figurarum sphæricarum utimur.

§. 4.

Ex angulis axium coordinatis duorum circulorum maximorum inveniendus est angulus horum circulorum.

In fig. 2. alter circulus axes in punctis A et B, alter in punctis A′ et B′ secet, linea cardinalis secetur in punctis C et C′; anguli coordinati alterius circuli sint x et y, alterius x′ et y′; sit denique $< ACX = z$, $A'C'X = z'$, $CY = \alpha$, $C'Y = \alpha'$, $< CRC' = d$. (Pro tang z et tang z′ non posita sunt z et z′).

Tum est: $\cos d = \sin z \,.\, \sin z' \,.\, \cos(\alpha - \alpha') + \cos z \,.\, \cos z' = \cos z \,.\, \cos z' (1 + \operatorname{tang} z \,.\, \operatorname{tang} z' \,.\, \cos(\alpha - \alpha'))$, et quum sit: $\cos(\alpha - \alpha') = \cos\alpha \,.\, \cos\alpha' + \sin\alpha \,.\, \sin\alpha' =$ $= \frac{x - y\cos v}{\operatorname{tang} z} \cdot \frac{x' - y'\cos v}{\operatorname{tang} z'} + \frac{y \sin v}{\operatorname{tang} z} \cdot \frac{y' \sin v}{\operatorname{tang} z'} = \frac{x \,.\, x' + y \,.\, y' - (x \,.\, y' + y \,.\, x')\cos v}{\operatorname{tang} z \,.\, \operatorname{tang} z'}$, habemus formulam:

$$\cos d = \frac{1 + x \,.\, x' + y \,.\, y' - (x \,.\, y' + y \,.\, x')\cos v}{\sqrt{[1 + x^2 + y^2 - 2x \,.\, y \,.\, \cos v]} \,.\, \sqrt{[1 + x'^2 + y'^2 - 2 \,.\, x' \,.\, y' \cos v]}},$$

ex qua sequuntur:

$$\sin d = \sqrt{\frac{(x' - x)^2 - 2(x' - x)(y' - y)\cos v + (y' - y)^2 + (xy' - yx')^2 \,.\, \sin^2 v}{(1 + x^2 + y^2 - 2 \,.\, x \,.\, y\cos v)(1 + x'^2 + y'^2 - 2 \,.\, x' \,.\, y' \,.\, \cos v)}}$$

$$\operatorname{tang} d = \frac{\sqrt{[(x' - x)^2 - 2(x' - x)(y' - y)\cos v + (y' - y)^2 + (x \,.\, y' - y \,.\, x')^2 \sin^2 v]}}{1 + x \,.\, x' + y \,.\, y' - (xy' + yx')\cos v}$$

Si in his formulis quantitates x, y, d constantes, x′, y′ vero variabiles sunt, circulus maximus $A'B' = (x', y')$ circulum minorem circumvelat, cujus radius sphæricus est $\frac{\pi}{2} - d$.

Eædem formulæ, si $v = \frac{\pi}{2}$ est, transeuns in simpliciores:

$$\cos d = \frac{1 + x \,.\, x' + y \,.\, y'}{\sqrt{(1 + x^2 + y^2)} \,.\, \sqrt{(1 + x'^2 + y'^2)}}; \quad \sin d = \sqrt{\frac{(x' - x)^2 + (xy' - yx')^2 + (y' - y)^2}{(1 + x^2 + y^2)(1 + x'^2 + y'^2)}};$$

$$\operatorname{tang} d = \frac{\sqrt{[(x' - x)^2 + (xy' - yx')^2 + (y' - y)^2]}}{1 + x \,.\, x' + y \,.\, y'}.$$

Ea conditione, ut angulus $d = \frac{\pi}{2}$ sit, habemus æquationem conditonalem:

$$1 + x \,.\, x' + y \,.\, y' - (xy' + yx')\cos v = o,$$

quæ transformari potest in hac:

$$1 + (x - y\cos v)\, x' + (y - x\cos v)\, y' = o,$$

quæ est *æquatio puncti*.

§. 5.

In fig. 2. est ex §. 3.:

$$y \,.\, \sin v = z \,.\, \sin\alpha \text{ et } x \,.\, \sin v = z \,.\, \sin\beta, \text{ ergo:}$$

$$y = \frac{\sin\alpha}{\sin\beta} \,.\, x,$$

quæ altera est æquatio puncti et talis quidem puncti, quod in linea cardinali situm est eoque efficitur, ut circulus maximus variabilis (x, y) hanc lineam secet.

Si qua data est fractio $\frac{\sin\alpha}{\sin\beta} = K$, non autem α et β ipsa, inveniuntur α et β ex notis his formulis:

$$\tan\alpha = \frac{K\sin v}{K\cos v - 1};\ \tan\beta = \frac{\sin v}{K - \cos v};\ \sin\alpha = \frac{K\sin v}{\sqrt{(1 - 2K\cos v + K^2)}};$$

$$\sin\beta = \frac{\sin v}{\sqrt{(1 - 2K\cos v + K^2)}};\ \cos\alpha = \frac{K\cos v - 1}{\sqrt{(1 - 2K\cos v + K^2)}};\ \cos\beta = \frac{K - \cos v}{\sqrt{(1 - 2K\cos v + K^2)}}.$$

§. 6.

Aequatio puncti, quod efficitur duobus circulis maximis se secantibus, qui angulos axium dividunt. Sint (fig. 3.) YC et XC arcus horum circulorum, per puncum C ductus sit circulus maximus, qui axes secet in punctis a et b. Sit præterea:

$$\angle YCb = m;\ \angle CYX = A;\ \angle aYX = x;\ \angle CYb = r$$

$$\angle XCa = n;\ \angle CXY = B;\ \angle bXY = y;\ \angle CXa = s$$

Tum est: $\frac{\sin r}{\sin m} = \frac{\sin bC}{\sin Yb};\ \sin r = \frac{\sin m\ .\ \sin bC}{\sin Yb}$,

$$\frac{\sin bC}{\sin Xb} = \frac{\sin(B - y)}{\sin n};\ \sin bC = \frac{\sin(B - y)}{\sin n}\ .\ \sin Xb,$$

ergo: $\sin r = \frac{\sin m\ .\ \sin(B - y)}{\sin n}\ .\ \frac{\sin Xb}{\sin Yb}$.

Si, ut jam antea fecimus, Xb per q significamus, est $\sin Yb = \sin q\ .\ \sin y$, ergo $\frac{\sin Xb}{\sin Yb} = \frac{1}{\sin y}$. Hoc valore substituto habemus, quum præterea sit $\sin r = \cos A$,

$$\frac{\sin n}{\sin m} = \frac{\sin(B - y)}{\sin y\ .\ \cos A}\ .\ .\ .\ .\ .\ (1)$$

Deinde est $\frac{\sin s}{\sin n} = \frac{\sin Ca}{\sin Xa};\ \sin s = \frac{\sin n\ .\ \sin Ca}{\sin Xa}$,

$$\frac{\sin Ca}{\sin Ya} = \frac{\sin(A - x)}{\sin m};\ \sin Ca = \frac{\sin(A - x)}{\sin m}\ .\ \sin Ya.$$

Est vero $\sin XA = \sin Ya\ .\ \sin x$. Habemus igitur, quum sit $\sin s = \cos B$,

$$\frac{\sin m}{\sin n} = \frac{\sin(A - x)}{\sin x\ .\ \cos B}\ .\ .\ .\ .\ .\ (2)$$

Si æquationes (1) et (2) multiplicando conjunguntur, fit:

$$\frac{\sin(B - y)}{\sin y\ .\ \cos A}\ .\ \frac{\sin(A - x)}{\sin x\ .\ \cos B} = 1,$$ hincque transformatione:

$$\frac{\tan x}{\tan A} + \frac{\tan y}{\tan B} = 1,$$

vel, si x pro tang x, y pro tang y, A pro tang A, B pro tang B ponuntur,

$$\frac{A}{x} + \frac{B}{y} = 1.$$

Hæc æquatio puncti C, si ad formam respicis, eodem modo, quo æquatio $y = \frac{\sin \alpha}{\sin \beta} . x$ in §. 5. cum notis æquationibus linearum rectarum, quæ inveniuntur in geometria analytica plana, omnino collabuntur.

Si punctum C cum linea cardinali producta arcu $Cc = \frac{\pi}{2}$ conjungimus et arcum cY per literam γ, arcum cX per literam λ, $<$ XcC per literam z obsignamus, est:

$$\text{tang } z = \text{tang } (\pi - A) \cos \gamma = - \text{tang } A . \cos \gamma; \text{ tang } A = - \frac{\text{tang } z}{\cos \gamma}.$$

Deinde $\text{tang } z = \text{tang } B . \cos \lambda; \text{ tang } B = \frac{\text{tang } z}{\cos \lambda}.$

His valoribus substitutis æquatio puncti C transit in:

$$y . \cos \lambda - x . \cos \gamma = \text{tang } z.$$

Postremo si arcus aY et bX cum linea cardinali producta conjunguntur arcubus $ec = dc = \frac{\pi}{2}$ et si obsignamus $<$ ecY per u, $<$ dcX per w, est

$$\text{tang } u = - x . \cos \gamma; \text{ tang } w = y . \cos \lambda.$$

Habemus igitur formulam

$$\text{tang } u + \text{tang } w = \text{tang } z.$$

§. 7.

Si æquatio generalis puncti alicujus hoc modo scricitur: $ax + b . y + c = o$, est $\text{tang } A = \frac{-c}{a}$; $\text{tang } B = \frac{-c}{a}$, et quum sit $\frac{\text{tang } A}{\text{tang } B} = - \frac{\cos \lambda}{\cos \gamma}$, habemus

$$\frac{\cos \lambda}{\cos \gamma} = - \frac{b}{a}.$$

Situs igitur verticis c anguli z non pendet a quantitate constante c æquationis $ax + by + c = o$.

Ex formula $\frac{\cos \lambda}{\cos \gamma} = - \frac{b}{a}$ deducitur:

$$\text{tang } \gamma = - \frac{b + a \cos v}{a \sin v}; \text{ tang } \lambda = \frac{a + b \cos v}{b . \sin v}.$$

Si obsignamus arcum YC per α, XC per β, invenitur

$$\text{tang } \alpha = \frac{\sin v \sqrt{(a^2 + c^2)}}{b + a \cos v}; \text{ tang } \beta = \frac{\sin v \sqrt{(b^2 + c^2)}}{a + b \cos v}.$$

Mutuus igitur situs angulorum XCY, CYX, CXY etiam a quantitate constante c pendet. Quantitas anguli z exprimitur per formulam

$$\text{tang } z = \pm \frac{c \sin v}{\sqrt{(a^2 + 2ab \cos v + b^2)}}.$$

Si quantitates a, b inducimus, est æquatio puncti C

$$y = -\frac{b + a \cos v}{a + b \cos v} \cdot x.$$

Anguli axium coordinati circuli maximi, cujus quadrans est Cc, sunt

$$x = -\frac{c \cdot (a + b \cos v)}{a^2 - b^2} \text{ et } y = \frac{c \cdot (b + a \cos v)}{a^2 - b^2}.$$

Si arc tang (p) et arc tang (q) quantitates coordinatæ circuli alicujus maximi sunt, æquatio centri ejus sphærici est

$$1 + (p - q \cos v)\, y + (q - p \cos v)\, y = o,$$

et si hanc æquationem puncti comparamus cum æquatione generali $ax + by + c = o$, manifestum est, necesse esse, ut sit

$$p - q \cos v = \frac{a}{c};\ q - p \cos v = \frac{b}{c},$$

ex quibus habebis

$$p = \frac{a + b \cos v}{c \cdot \sin^2 v};\ q = -\frac{b + a \cos v}{c \cdot \sin^2 v}.$$

Ergo, si data est æquatio puncti $a \cdot x + b \cdot y + c = o$, tangentes angulorum axium coordinatorum illius circuli maximi, qui ex hoc puncto describitur, sunt

$$\frac{a + b \cos v}{c \sin^2 v} \text{ et } \frac{b + a \cos v}{c \sin^2 v}.$$

Quum ex formulis supra commemoratis sit $\frac{q}{p} = \frac{b + a \cos v}{a + b \cos v}$, æquatio puncti C præterea est

$$y = -\frac{q}{p} \cdot x.$$

§. 8.

Eodem modo, quo in geometria analytica plana æquatio lineæ rectæ per punctum (x, y) datum ductæ invenitur, hic invenis æquationem

$$x - x' = \alpha\, (y - y')$$

puncti, quod situm est in circulo maximo, cujus angulorum axium coordinatorum tangentes sunt x' et y'. Quantitas α determinata non est, quod mirum esse non potest, quum in circulo maximo (x', y') innumerabilia puncta sita esse possint.

Simili modo invenitur æquatio

$$x - x' = \frac{y'' - y'}{x'' - x'} \cdot (y - y')$$

puncti, quod in duobus circulis maximis (x', y') et (x'', y'') situm est. Si in hac æquatione ponimus $x = x'$, fit $y = y'$, et si ponimus $x = x''$, fit etiam $y = y''$, unde elucet, punctum, cujus æquatio illa sit, re vera in utroque circulo maximo situm esse. Facile intelligitur, hanc æquationem etiam hoc modo scribi posse:

$$y - y'' = \frac{y'' - y'}{x'' - x'} (x - x''), \text{ vel etiam:}$$

$$(y'' - y') x - (x'' - x') y = y'' . x' - x'' . y'.$$

Utimur hac formula ad inveniendam æquationem puncti, quod a duobus punctis datis quadrante distet. Sint

$$ax + by + c = o \text{ et } a'x + b'y + c' = o$$

æquationes punctorum datorum, sit porro MM circulus maximus ex puncto, cujus æquatio est $ax + by + c = o$, descriptus, M'M' vero circulus maximus ex puncto, cujus æquatio est $a'x + b'y + c' = o$, descriptus. Punctum intersectionis horum circulorum ab utroque puncto dato quadrante distabit. Quum vero hoc punctum in duobus circulis maximis situm sit, æquatio ejus habebit formam

$$(y'' - y') x - (x'' - x') y = y'' . x' - x'' . y'.$$

In hac æquatione ponendum est ex §. 7.

$$\frac{a + b \cos v}{c \sin^2 v} \text{ pro } x'; \quad \frac{a' + b' \cos v}{c' . \sin^2 v} \text{ pro } x''; \quad \frac{b + a \cos v}{c . \sin^2 v} \text{ pro } y'; \quad \frac{b' + a' \cos v}{c' . \sin^2 v} \text{ pro } y'';$$

His quantitatibus substitutis æquatio invenienda est

$$Ax + B . y + C = o$$

in qua est:

$$A = b'c - bc' + (ca' - ac') \cos v; \; B = ac' - ca' + (bc' - cb') \cos v; \; C = a'b - b'a.$$

Si sunt arc tang (p) et arc tang (q) quantitates coordinatæ ejus circuli maximi, cujus centrum sphæricum est punctum, cujus æquatio est $Ax + By + C = o$, est:

$$p = \frac{A + B \cos v}{C \sin^2 v}; \; q = \frac{B + A \cos v}{C \sin^2 v}, \text{ vel } p = \frac{b'c - c'b}{a'b - ab'}; \; q = \frac{ac' - ca'}{a'b - ab'}.$$

Præterea est $a . p + b . q + c = o$ et $a'p + b'q + c' = o$, quo apparet, circulum maximum = (arc tang (p), arc tang (q)) per utrumque punctum datum ire, id, quod etiam ex aliis notum est.

§. 9.

Facile est, distantiam duorum punctorum ex æquationibus datis eorum constituere. Aequationes datæ sint $ax + by + c = o$ et $a'x + b'y + c' = o$.

Angulus, quem circuli maximi ex his punctis descripti intersectione faciunt, distantiæ punctorum aut æqualis, aut ejus spplementum erit. Si supponimus, angulum illum distantiæ esse æqualem, et tum puncta data per M et M', angulum distantiæ æqualem per d,

circulum ex puncto M descriptum per (x', y'), circulum ex M' descriptum per (x'', y'') obsignamus, est

$$\cos MM' = \cos d = \frac{1 + x' . x'' + y' . y'' - (x'y'' + y'x'') \cos v}{\sqrt{(1 + x'^2 + y'^2 - 2x'y' \cos v)} . \sqrt{(1 + x''^2 + y''^2 - 2x'' . y'' \cos v)}}.$$

Quum vero sit:

$$x' = \frac{a + b \cos v}{c \sin^2 v}; \; y' = \frac{b + a \cos v}{c . \sin^2 v}; \; x'' = \frac{a' + b' \cos v}{c' . \sin^2 v}; \; y'' = \frac{b' + a' \cos v}{c' \sin^2 v}, \text{ est}$$

$$1 + x' . x'' + y' . y'' - (x'y'' + y' . x'') \cos v = \frac{a . a' + b . b' + c . c' . \sin^2 v + (ab' + ba') \cos v}{c . c' . \sin^2 v},$$

$$1 + x'^2 + y'^2 - 2x' . y' \cos v = \frac{a^2 + b^2 + c^2 \sin^2 v + 2ab \cos v}{c^2 \sin^2 v},$$

$$1 + x''^2 + y''^2 - 2x'' . y'' \cos v = \frac{a'^2 + b'^2 + c'^2 \sin^2 v + 2a'b' \cos v}{c'^2 \sin^2 v}.$$

Habemus igitur, his valoribus substitutis, hanc formulam generalem:

$$\cos MM' = \frac{aa' + bb' + c . c' \sin^2 v + (ab' + ba') \cos v}{\sqrt{(a^2 + b^2 + c^2 \sin^2 v + 2abc \cos v)} . \sqrt{(a'^2 + b'^2 + c'^2 \sin^2 v + 2a'b' \cos v)}},$$

ex qua facile deduces

$$\sin MM' = \sin v \sqrt{\frac{(ab' - ba')^2 + (ac' - ca')^2 + (bc' - cb')^2 + 2(ac' - ca')(bc' - cb') \cos v}{(a^2 + b^2 + c^2 \sin^2 v + 2ab \cos v)(a'^2 + b'^2 + c'^2 \sin^2 v + 2a'b' \cos v)}}.$$

Si linea cardinalis quadrans est circuli maximi, hæ formulæ fiunt

$$\cos MM' = \frac{aa' + bb' + c . c'}{\sqrt{(a^2 + b^2 + c^2)} . \sqrt{(a'^2 + b'^2 + c'^2)}}; \; \sin MM' = \sqrt{\frac{(ab' - ba')^2 + (ac' - ca')^2 + (bc' - cb')^2}{(a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2)}},$$

$$\tan MM' = \frac{\sqrt{[(ab' - ba')^2 + (ac' - ca')^2 + (bc' - cb')^2]}}{aa' + b . b' + c . c'}.$$

Ea addita conditione, ut sit distantia MM' quadrans circuli maximi, habemus æquationem conditionalem

$$aa' + bb' + c . c' \sin^2 v + (ab' + ba') \cos v = 0, \text{ quæ, si est } v = \frac{\pi}{2}, \text{ fit}$$

$$aa' + bb' + c . c' = 0.$$

§. 10.

(Fig. 4.) In circulo maximo $(AB) = (m, n)$ dato inveniatur punctum, quod a puncto M, cujus æquatio $ax + by + c = 0$ data sit, quadrante distet.

Sit $(ab) = (t, u)$ circulus maximus ex puncto M descriptus. Tum est

$$t = \frac{a + b \cos v}{c . \sin^2 v}; \; u = \frac{b + a \cos v}{c . \sin^2 v}.$$

Si circuli maximi $(AB) = (m, n)$ et $ab = (t, u)$ producuntur, dum se secant, intersectionis punctum P a puncto M dato quadrante distat. Deinde si angulum APa per α,

MPA per β obsignamus, est $\alpha - \beta = \frac{\pi}{2}$, ergo $\cos\alpha = -\sin\beta$. Est autem ex §. 4.

$$\cos\alpha = \frac{1 + m(t - u\cos v) + n(u - t\cos v)}{\sqrt{(1 + t^2 + u^2 - 2tu\cos v)} \cdot \sqrt{(1 + m^2 + n^2 - 2mn\cos v)}}.$$

Præterea est $t - u\cos v = \frac{a}{c}$; $u - t\cos v = \frac{b}{c}$; $\sqrt{(1+t^2+u^2-2tu\cos v)} = \frac{\sqrt{(a^2+b^2+c^2\sin^2 v+2ab\cos v)}}{c\sin v}$.

His substitutis valoribus habemus

$$-\sin\beta = \cos\alpha = \frac{(ma + nb + c)\sin v}{\sqrt{(1 + m^2 + n^2 - 2mn\cos v)} \cdot \sqrt{(a^2 + b^2 + c^2\sin^2 + 2ab\cos v)}};$$

Aequatio puncti P est

$$y - n = \frac{u - n}{t - m} \cdot (x - m), \text{ vel}$$

$$y - n = \frac{b + a\cos v - nc\sin^2 v}{a + b\cos v - mc\sin^2 v} \cdot (x - m).$$

Si linea cardinalis quadrans est, habemus æquationem simpliciorem

$$y - n = \frac{nc - b}{mc - a} \cdot (x - m).$$

§. 11.

Si in fig. 4. punctum datum M cum puncto P invento et præterea cum aliquo alio puncto W circuli maximi AB = (m, n) conjungitur et si obsignamus arcum MW per ϱ, angulum MWA per φ, est, quum $MP = \frac{\pi}{2}$ sit,

$$\sin\beta = \sin\varphi \cdot \sin\varrho;\ \sin\varphi = \frac{\sin\beta}{\sin\varrho}$$

Hac relatione usi facile aliud problema solvere possumus. „Sint data duo puncta, inveniatur æquatio tertii, quod distantiam datorum in duas æquales partes dividat.“

$ax + by + c = o$ et $a'x + b'y + c' = o$ sint æquationes punctorum datorum M et M'. Punctum quærendum in arcu MM' situm est.

Si per hoc punctum, quod per literam o obsignamus, circulum maximum ducimus, ejusque punctum, quod a puncto M quadrante distat per literam R, id autem, quod a puucto M' quadrante distat, per literam Q obsignamus, et si tum puncta M et R, M' et Q arcubus circulorum maximorum conjungimus, habemus duo triangula rectilatera, MRo et M'Qo. Sit $\angle$ MRo $= \beta$; M'Qo $= \beta'$; RoM $= \varphi$, QoM' $= \varphi'$; Mo $=$ M'o $= \varrho$. Tum est

$$\sin\varphi = \frac{\sin\beta}{\sin\varrho};\ \sin\varphi' = \frac{\sin\beta'}{\sin\varrho} \text{ et præterea } \sin\varphi = \sin\varphi'.$$

Formulis in §. 10. deductis hic substitutis fit

$$\sin\varphi = -\frac{\sin v}{\sin\varrho} \cdot \frac{ax + by + c}{\sqrt{(1 + x^2 + y^2 - 2xy\cos v)} \cdot \sqrt{(a^2 + b^2 + c^2\sin^2 v + 2ab\cos v)}}$$

$$\sin\varphi' = -\frac{\sin v}{\sin\varrho} \cdot \frac{a'x + b'y + c'}{\sqrt{(1+x^2+y^2-2xy\cos v)} \cdot \sqrt{(a'^2+b'^2+c'^2\sin^2 v+2a'b'\cos v)}},$$

unde fieri quisque videt

$$\frac{ax + by + c}{\sqrt{(a^2 + b^2 + c^2\sin^2 v + 2ab\cos v)}} = \pm \frac{a'x + b'y + c'}{\sqrt{(a'^2 + b'^2 + c'^2\sin^2 v + 2a'b'\cos v)}}.$$

Hæc est æquatio puncti distantiam duorum punctorum, quorum æquationes $ax + by + c = o$ et $a'x + b'y + c' = o$ datæ sunt, in duas æquales partes dividentis. Quantitates x, y, quæ in hac æquatione sunt, tangentes angulorum axium coordinatorum circuli maximi per punctum quærendum ducti sunt.

Signum ($\pm$) ante latus dexterum æquationis significat, æquationem pertinere ad duo puncta, quorum vero alter arcum MM' ipsum, alter illum arcum, qui MM' ad totum circulum maximum supplet, in duas æquales partes dividit.

Si non postulatur, ut arcus MM' in duas *æquales* partes dividatur, sed ut relatio inter ejus partes $\frac{\sin\varrho}{\sin\varrho'}$ sit æqualis fractioni datæ $\frac{m}{n}$, habemus

$$\pm \frac{\sin v}{\sin\varrho} \cdot \frac{ax + by + c}{\sqrt{(1 + x^2 + y^2 - 2xy\cos v)} \cdot \sqrt{(a^2 + b^2 + c^2\sin^2 v + 2ab\cos v)}} =$$

$$= \frac{\sin v}{\sin\varrho'} \cdot \frac{a'x + b'y + c'}{\sqrt{(1 + x^2 + y^2 - 2xy\cos o)} \cdot \sqrt{(a'^2 + b'^2 + c'^2\sin^2 v + 2a'b'\cos v)}},$$

vel $\pm \frac{m}{n}\sqrt{\frac{a^2 + b^2 + c^2\sin^2 v + 2ab\cos v}{a'^2 + b'^2 + c'^2\sin^2 v + 2a'b'\cos v}} = \frac{ax + by + c}{a'x + b'y + c'}$, vel, si ponimus

k pro $\frac{m}{n}\sqrt{\frac{a^2 + b^2 + c^2\sin^2 v + 2ab\cos v}{a'^2 + b'^2 + c'^2\sin^2 v + 2a'b'\cos v}}$,

$$ax + by + c = \pm k \, . \, (a'x + b'y + c').$$

Signum indeterminatum ($\pm$) ex eo proficiscitur, quod arcus MM' et interne et externe ex quantitate $\frac{m}{n}$ dividi potest. Sub formam $ax + by + c = o$ redacta, æquatio est

$$(a - k \, . \, a')\,x + (b - k \, . \, b')\,y + (c - k \, . \, c') = o.$$

In fig. 5. sit $ax + by + c = o$ æquatio puncti M, $a'x + b'y + c' = o$ æquatio punti M', sit præterea circulus maximus $qq' = (x', y')$ constans, circulus maximus $pp' = (x, y)$ variabilis. Tum est

$$ax + by + c = k\,(a'x + b'y + c')$$

æquatio puncti a et eodem modo

$$ax' + by' + c = k\,(a'x' + b'y' + c')$$

æquatio puncti b, si k est quantitas constans ex formula supra commemorata invenienda. Dividendo nanciscimur

$$\frac{a \, . \, x + by + c}{a \, . \, x' + b \, . \, y' + c} = -\frac{a'x + b'y + c'}{a'x' + b'y' + c'}, \text{ vel}$$

$$\frac{ax + by + c}{ax' + by' + c} + \frac{a'x + b'y + c'}{a'x' + b'y' + c'} = o,$$

æquationem puncti a. Ex analogia circulus maximus qq′ nominari potest *polus puncti polaris* a et eodem modo circ. max. pp′ polus puncti polaris b.

Hæc æquatio puncti polaris transformari potest in hanc:

$$2b \cdot b'yy' + (a'b + ab')(x'y + yx') + 2aa'xx' + (bc' + cb')(y+y') + (ac' + ca')(x+x') + 2cc' = o.$$

§. 12.

Si sunt $y - u = \alpha(x - t)$ et $y - u = \alpha'(x - t)$ æquationes duorum punctorum in circulo maximo (t, u) sitorum, æquatio, quae multiplicatione harum æquationum procedit, est æquatio systematis duorum illorum punctorum. Est hæc æquatio:

$$(y - u - \alpha(x - t))(y - u - \alpha'(x - t)) = o,$$

quæ, ex y et x ordinata hanc habet formam generalem

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + G = o.$$

Si data est æquatio hujus formæ æquatio systematis duorum punctorum, quantitates α et α' inveniuntur ex his formulis

$$\alpha = \frac{-B + \sqrt{(B^2 - AC)}}{A}; \; \alpha' = \frac{-B - \sqrt{(B^2 - AC)}}{A}.$$

Ad determinandas quantitates t et u facile invenies formulas

$$t = \frac{AE - BD}{B^2 - AC} = \frac{D^2 - AG}{AE - BD} = \frac{BG - DE}{CD - BE}; \; u = \frac{CD - BE}{B^2 - AC} = \frac{BG - DE}{AE - BD} = \frac{E^2 - CG}{CD - BE}.$$

Ex his formulis invenitur æquatio conditionalis, quæ esse debet, ut æquatio illa generalis æquatio systematis duorum punctorum esse possit. Est ea

$$A \cdot E^2 - 2BDE + C \cdot D^2 + G(B^2 - AC) = o.$$

Ad determinandam distantiam duorum punctorum adjutus formulis §. 9. facile invenies formulam hanc:

$$\text{tang } MM' = \frac{2 \sin v \sqrt{[A' + C' + G' - 2B' \cos v]}}{A + 2B \cos v + C + G \sin^2 v},$$

in qua est

$$A' = E^2 - CG; \; B' = BG - DE; \; C' = D^2 - AG; \; D' = CD - BE; \; E' = AE - BD;$$
$$G' = B^2 - AC.$$

Ut puncta M et M′ quadrante distent, valere debet æquatio conditionalis

$$A + 2B \cos v + C + G \sin^2 v = o,$$

quæ, si $v = \frac{\pi}{2}$ est, fit

$$A + C + G = o.$$

Adnot. De systemate duorum punctorum præter hæc plurima dicenda sunt, quæ hoc loco omittimus, quod ad evolvendas formulas fundamentales minoris sunt momenti.

§. 13.

Aequatio puncti polaris, cujus supra mentionem fecimus, etiam ex æquatione systematis duorum punctorum deduci potest. Sit $ax + by + c = o$ æquatio unius, $a'x + b'y + c' = o$ æquatio alterius puncti; æquatio systematis tum est

$$(ax + by + c)(a'x + b'y + c') = o, \text{ vel evoluta}$$

$$bb'y^2 + (ab' + ba')xy + aa'x^2 + (cb' + bc')y + (ac' + ca')x + cc' = o.$$

Si hanc æquationem cum æquatione generali $Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + G = o$ comparamus, esse debet

$$bb' = A;\ ab' + ba' = 2B;\ aa' = C;\ cb' + bc' = 2D;\ ac' + ca' = 2E;\ cc' = G.$$

Si hos valores in æquatione puncti polaris (§. 11.) substituimus, fit æquatio illa

$$2Ayy' + 2B(xy' + yx') + 2Cxx' + 2D(y + y') + 2E(x + x') + 2G = o, \text{ vel}$$

$$Ayy' + B(xy' + yx') + Cxx' + D(y + y') + E(x + x') + G = o,$$

quæ si, quum x' et y' quantitates constantes sint, ex y et x ordinatur, transit in

$$(Ay' + Bx' + D)y + (By' + Cx' + E)x + Dy' + Ex' + G = o$$

æquationem puncti polaris ex æquatione systematis duorum punctorum evolutam. Tangentes cyclici angulorum axium coordinatorum circuli maximi poli sunt x' et y'.

Si linea cardinalis polus est, est æquatio puncti polaris

$$Dy + Ex + G = o.$$

Si axis XV polus est, est æquatio puncti polaris

$$By + Cx + E = o.$$

Si axis altera est polus, habemus æquationem puncti polaris

$$Ay + Bx + D = o.$$

Easdem tres æquationes in §. 12. ad determinandas quantitates t et u, quibus situs circuli maximi per duo puncta systematis ducti datus est, invenies, et quum quantitates t et u, (hoc loco x et y) cuique trium æquationum satisfaciant, tria puncta polaria, quorum poli sunt latera trianguli quantitatum coordinatarum, in circulo maximo, cujus arcus est linea cardinalis, sita sunt.

De mutatione angulorum axium coordinatorum.

§. 14.

Situs lineæ cardinalis non mutetur, axes vero aliam directionem accipiant. Sint (fig. 5.) $XV = YV = \frac{\pi}{2}$ axes veteres, $X'V = Y'V = \frac{\pi}{2}$ axes novi; sit postremum $XY = v$,

$X'Y' = v'$, $YY' = \alpha$, $XX' = \beta$. Tangentes cyclici angulorum axium coordinatorum circuli maximi (AB) relati ad triangulum XVY sint x et y, ejusdem circuli maximi relati ad triangulum X'Y'V x' et y'; tangens cyclicus anguli ACY sit z. Si præterea arcus CY obsignatur litera m, arcus CX litera n, ergo arcus CY' per $(m + \beta)$, CX' per $(n + \alpha)$, est:

$$\sin m = \frac{y \sin v}{z}; \quad \cos m = \frac{x - y \cos v}{z}; \quad \sin (m + \beta) = \frac{y' \sin v'}{z};$$

$$\sin n = \frac{x \sin v}{z}; \quad \cos n = \frac{y - x \cos v}{z}; \quad \sin (n + \alpha) = \frac{x' \sin v'}{z};$$

$$\cos (m + \beta) = \frac{x' - y' \cos v'}{z}; \quad \cos (n + \alpha) = \frac{y' - x' \cos v'}{z},$$

unde sequitur

$$\frac{x'}{z} = \frac{\sin (n + \alpha)}{\sin v'} = \frac{\sin n \cos \alpha + \cos n \sin \alpha}{\sin v'},$$

vel substitutis valoribus functionum sin n et cos n

$$x' = \frac{y \sin \alpha + x \sin (v - \alpha)}{\sin v'} \; \ldots\ldots\ldots \; 1.$$

Simili modo invenies

$$y' = \frac{x \sin \beta + y \sin (v - \beta)}{\sin v'} \; \ldots\ldots\ldots \; 2.$$

$$y = \frac{y' \sin (v' + \beta) - x' \sin \beta}{\sin v} \; \ldots\ldots \; 3.$$

$$x = \frac{x' \sin (v' + \alpha) - y' \sin \alpha}{\sin v} \; \ldots\ldots\ldots \; 4.$$

Formulis 1. et 2. novi anguli coordinati ad veteres, formulis 3. et 4. veteres ad novos reducuntur. Omnibus his formulis maxima est similitudo cum formulis geometriæ analyticæ planæ, quæ existunt, si in systemate linearum coordinatarum punctum, ex quo proficiscuntur axes coordinati, idem remanet, axes vero aliam accipiunt directionem.

§. 15.

Problema mutandorum angulorum axium coordinatorum multo difficilius fit, si non solum axibus aliæ directiones dantur, sed præterea etiam linea cardinalis alium accipit situm.

In fig. 6. sit XYV systema vetus, X'Y'V' vero novum, sint x et y tangentes cyclici angulorum coordinatorum circuli maximi (AA'B'B) relati ad systema vetus, x' et y' tangentes cyclici ang. coord. ejusdem circuli relati ad systema novum; deinde obsignetur angulus YHY' per e, arcus HX per m, HY per n, HX' per m' HY' per n'; m — n per v, m' — n' per v'. Arcus HE, quo punctum intersectionis linearum cardinalium conjungitur cum circulo maximo (AA'B'B) sit quadrans. Tum est:

$$\sin GX = \frac{y \sin v}{\text{tang } XGB}, \quad \cos GX = \frac{x - y \cos v}{\text{tang } XGB}.$$

Quum sit $HG = m - GX$, est

$$\sin HG = \sin m \cos GX - \cos m \sin GX = \frac{x \sin m - y \cos v \sin m - y \sin v \cos m}{\text{tang } XGB}, \text{ vel}$$

$$\sin HG = \frac{x \sin m - y \sin (v + m)}{\text{tang } XGB} = \frac{y \sin (v + m) - x \sin m}{\text{tang } EGH}.$$

Deinde

$$\cos GH = \cos m \cos GX + \sin m \sin GX = \frac{x \cos m - y \cos v \cos m + y \sin v \sin m}{\text{tang } XGB},$$

$$\text{vel } \cos GH = \frac{x \cos m - y \cos (v + m)}{\text{tang } XGB} = \frac{y \cos (v + m) - x \cos m}{\text{tang } EGH}.$$

Est etiam

$$\sin GEH = \sin EGH \sin HG = \cos EGH \,.\, (y \sin (v + m) - x \sin m).$$

Præterea habemus

$$\text{tang } GHE = \text{tang } XGB \cos HG = x \cos m - y \cos (v + m),$$

et, quum sit $GHE = e + G'HE$, ergo $\text{tang } GHE = \text{tang } (e + G'HE)$,

$$x \cos m - y \cos (v + m) = \frac{\text{tang } e + \text{tang } G'HE}{1 - \text{tang } e \text{ tang } G'HE}, \text{ vel}$$

$$x \cos m - y \cos (v + m) = \frac{\sin e + \cos e \text{ tang } G'HE}{\cos e - \sin e \text{ tang } G'HE}.$$

Sed quum sit

$\text{tang } G'HE = x' \cos m' - y' \cos (v' + m')$, habemus:

$$x \cos m - y \cos (v + m) = \frac{\sin e + x' \cos m' \cos e - y' \cos (v' + m') \cos e}{\cos e - x' \cos m' \sin e + y' \cos (v' + m') \sin e} \quad . \; . \; . \; 1.$$

Deinde est in systemate vetere

$$\sin GEH = \cos EGH \,.\, [y \sin (v + m) - x \sin m],$$

in systemate novo autem

$$\sin GEH = \cos EG'H \,.\, [y' \sin (v' + m') - x' \sin m'], \text{ ergo}$$

$$y \sin (v + m) - x \sin m = \frac{\cos EG'H}{\cos EGH} \,.\, [y' \sin (v' + m') - x' \sin m'].$$

Factor $\frac{\cos EG'H}{\cos EGH}$ aliter exprimi potest. Est enim

$$\cos EGH = - \cos GHE \cos GEH \text{ et } \cos EG'H = - \cos G'HE \cos GEH,$$

ergo $\frac{\cos EG'H}{\cos EGH} = \frac{\cos G'HE}{\cos GHE}$, vel, quum sit

$$\cos GHE = \cos (e + G'HE) = \cos e \cos G'HE - \sin e \sin G'HE,$$

$$\frac{\cos EG'H}{\cos EGH} = \frac{1}{\cos e - \sin e \text{ tang } G'HE}, \text{ vel quum sit}$$

$$\text{tang } G'HE = x' \cos m' - y' \cos (v' + m'),$$

$$\frac{\cos EG'H}{\cos EGH} = \frac{1}{\cos e - x' \cos m' \sin e + y' \cos (v' + m') \sin e}.$$

Hoc valore substituto habemus

$$y \sin (v + m) - x \sin m = \frac{y' \sin (v' + m') - x' \sin m'}{\cos e - x' \cos m' \sin e + y' \cos (v' + m') \sin e} \quad \ldots \; 2.$$

Ex æquationibus 1. et 2. facile deduces has formulas

$$x \sin v = \frac{\sin e \sin(v+m) + [\cos m' \cos e \sin(v+m) - \sin m' \cos(v'+m')]x' + [\sin(v'+m') \cos(v+m) - \cos(v'+m') \cos e \sin(v+m)].y'}{\cos e - \cos m' \sin e \, x' + \cos (v' + m') \sin e \, y'},$$

$$y \sin v = \frac{\sin e \sin m + [\cos m' \cos e \sin m - \sin m' \cos m] \, x' + [\sin (v' + m') \cos m - \cos (v' + m') \cos e \sin m] \, y'}{\cos e - \cos m' \sin e \, x' + \cos (v' + m') \sin e \, y'},$$

quæ hanc formam generalem habent

$$x = \frac{p + qx' + ry'}{1 + \beta x' + \gamma y'}; \quad y = \frac{p' + q'x' + r'y'}{1 + \beta x' + \gamma y'}.$$

Ceteras formulas fundamentales geometriæ analyticæ sphæricæ hoc loco communicare in animo non est. Has brevissimo fortasse tempore simul cum curvis nonnullis sphæricis in usu angulorum axium coordinatorum tractatis in lucem edemus.

V i t a.

Ignatius Philippus Renvers natus sum Telgeti exeunte anno hujus saeculi XXIV patre *Bernhardo*, quem ante quinque jam annos morte mihi abreptum lugeo, matre *Clara* e gente *Neuhausiana*, quam superstitem vereor semperque verebor. Fidei addictus sum catholicae.

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Omnibus his viris optime de me meritis ceterisque, qui studia mea sublevaverunt, gratias ago quam maximas possum.

Tabula figurarum.

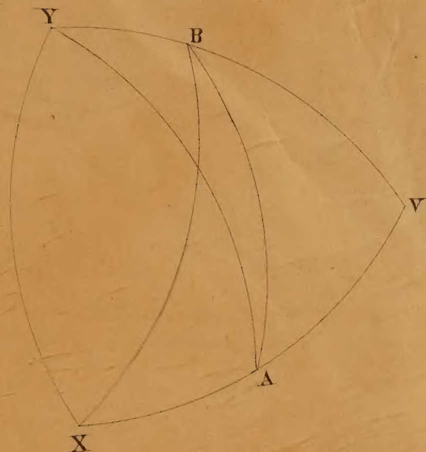


Fig. 1.

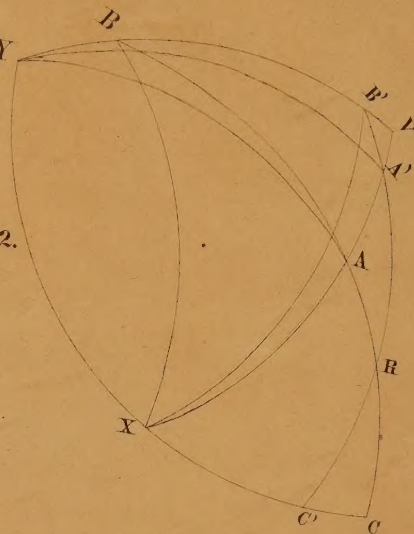


Fig. 2.

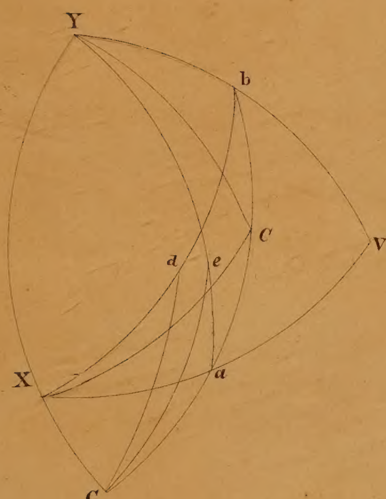
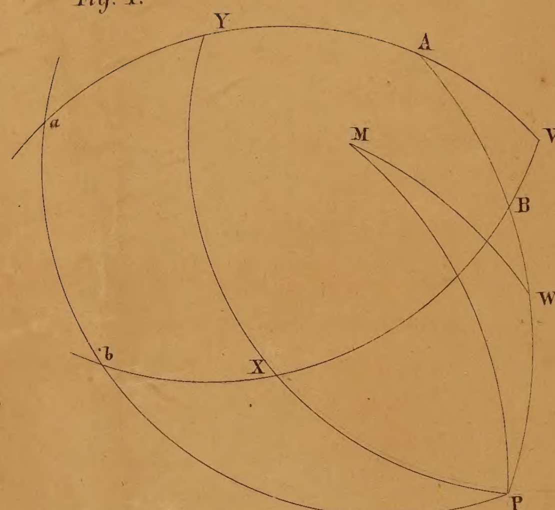


Fig. 3.

Fig. 4.



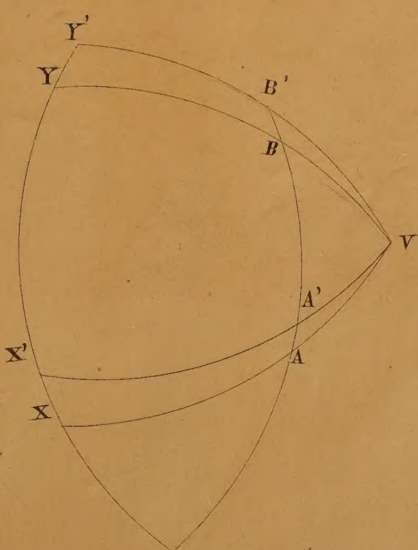


Fig. 5.

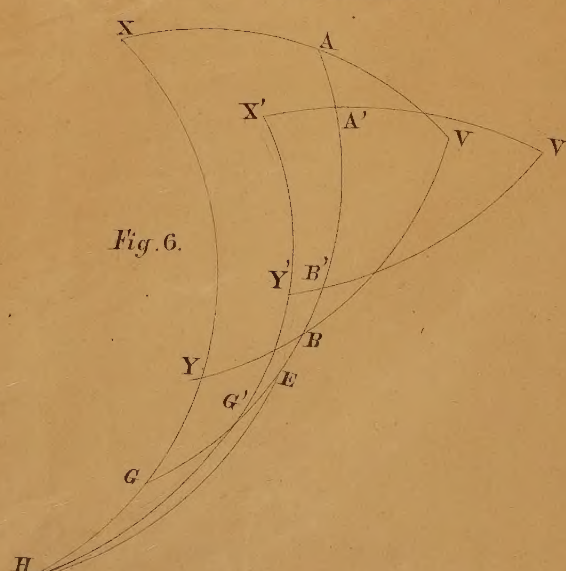


Fig. 6.